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Spectral transformations, self-similar reductions and orthogonal polynomials

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Abstract. We study spectral transformations in the theory of orthogonal polynomials which are similar to Darboux transformations for the Schrödinger equation. Linear transformations of the Stieltjes function with coefficients that are rational in the argument are constructed as iterations of the Christoffel and Geronimus transformations. We describe a characteristic property of semi-classical orthogonal polynomials (SCOP) on the uniform and the exponential lattice; namely, that all these polynomials can be obtained through simple quasi-periodic and q-periodic (self-similar) closures of the chain of linear spectral transformations. In the self-similar setting, a characterization of the Laguerre–Hahn polynomials on linear and q-linear lattices is obtained by considering rational transformations of the Stieltjes function generated by transitions to the associated polynomials.

1. Introduction

Suppose we have an eigenvalue problem for a linear differential, difference, or differential–difference operator L, $L\psi = \lambda\psi$. Let us take an operator D of similar nature and demand the existence of an intertwining relation $DL = \tilde{L}D$, where $\tilde{L}$ has the same form as L with different coefficients in front of the differential or difference operators. The formal solutions of the eigenvalue problem $\tilde{L}\tilde{\psi} = \lambda\tilde{\psi}$ are of course given by $\tilde{\psi} = D\psi$, but the existence of boundary conditions and of the zero modes of D makes it non-trivial to find the full spectrum of $\tilde{L}$ from the known properties of L. We call the transformation of ψ into $\tilde{\psi}$ *spectral* if the spectral characteristics of $\tilde{L}$ can be found exactly from those of L. This definition does not provide a fully determined class of spectral transformations but we shall use in this paper a set of simple transformations which is rich enough to cover special classes of orthogonal polynomials.

The best known example of such a transformation is given by the Darboux transformation, which has found wide applications in quantum mechanics and in the theory of nonlinear integrable systems, see for example [16, 22]. Under certain restrictions it allows one to intertwine two Schrödinger operators whose spectra differ from each other by at most one discrete point. The discrete Darboux transformation (DDT) allows one to make similar spectral changes for the discrete Schrödinger equation or three-term recurrence relation for orthogonal polynomials

$$A_n\psi_{n+1} + B_n\psi_n + C_n\psi_{n-1} = \Lambda\psi_n \tag{1.1}$$

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where A_n, B_n, C_n are some real coefficients, ψ_n is an unknown discrete wavefunction and Λ is a spectral parameter. The discrete index n can vary either from $-\infty$ to ∞ (unrestricted problem), from 0 to ∞ (restricted problem), or even take complex values. There are also special classes of finite-dimensional problems, when $0 \leqslant n \leqslant N$.

Using the freedom associated with the normalization of ψ_n in equation (1.1), either A_n or C_n can be set equal to 1. We choose $A_n = 1$ and rewrite (1.1) in the form

$$L\psi \equiv \psi_{n+1} + b_n\psi_n + u_n\psi_{n-1} = \lambda\psi_n. \tag{1.2}$$

Various aspects of the DDT for (1.2) were discussed in [10, 16, 23] and other works (for a list of relevant references in the orthogonal polynomials literature, see [3, 6, 14, 26]). The explicit form of the DDT used in this paper is

$$\tilde{\psi}_n = D\psi_n \equiv \psi_n - \frac{\phi_n}{\phi_{n-1}}\psi_{n-1} \tag{1.3}$$

where ϕ_n is a special solution of the discrete Schrödinger equation (1.2) for the special value $\lambda = \mu$ of the spectral parameter. If $\psi_n(\lambda)$ is a general solution of (1.2) then $\tilde{\psi}_n(\lambda)$ is the general solution (for $\lambda \neq \mu$) of the equation

$$\tilde{L}\psi \equiv \tilde{\psi}_{n+1} + \tilde{b}_n\tilde{\psi}_n + \tilde{u}_n\tilde{\psi}_{n-1} = \lambda\tilde{\psi}_n \tag{1.4}$$

with the new recurrence coefficients

$$\tilde{u}_n = u_{n-1}\frac{\phi_n\phi_{n-2}}{\phi_{n-1}^2} \qquad \tilde{b}_n = b_n + \frac{\phi_{n+1}}{\phi_n} - \frac{\phi_n}{\phi_{n-1}}. \tag{1.5}$$

Consider an infinite chain of successive DDTs with parameters μ_j and the corresponding solutions of (1.2) ϕ_n^j, $j \in \mathbb{Z}$. Denote $A_n^j = \phi_n^j/\phi_{n-1}^j$. Then, evidently, equations (1.5) can be rewritten in the form

$$u_n^{j+1} = u_{n-1}^j\frac{A_n^j}{A_{n-1}^j} \qquad b_n^{j+1} = b_n^j + A_{n+1}^j - A_n^j. \tag{1.6}$$

These are the (non-isospectral) discrete-time Toda chain equations [25], where the variable j plays the role of the discrete evolution parameter ('time').

In [23] various similarity reductions of equations (1.6) have been considered. Let us recap on the corresponding definitions. There are obvious symmetries mapping the space of solutions of (1.6) onto itself—the shifts of j and n by some integers, the shifts of b_n^j by a constant, and the appropriate uniform scalings of u_n^j, b_n^j and A_n^j. One can consider a subclass of solutions which is invariant under the general combination of these discrete symmetry transformations with fixed parameters of shifts and scalings. Such solutions are called self-similar [22, 23]. The key property of the resulting recurrence coefficients (potentials) is that for them there exists a finite sequence of DDTs, the effect of which is equivalent to a simple affine transformation of the initial potentials (see later).

The transformation (1.3) was used in [25] to explicitly construct particular reflectionless potentials in the context of 'discrete quantum mechanics'. These potentials turned out to be closely connected to the special classes of the Askey–Wilson polynomials. In [26] it was shown that recurrence coefficients of the Askey–Wilson polynomials provide a simple solution of the discrete-time Toda chain equations (1.6).

We continue in this paper a systematic study of the DDT for orthogonal polynomials (OP), thus pursuing the work undertaken in [23–26]. There are in general two different types of DDT transforming OP into OP of the same degree; they bear the names of Christoffel and Geronimus. An essential clue to the whole consideration is provided by the observation that the class of semi-classical orthogonal polynomials (SCOP) and the Laguerre–Hahn polynomials [11, 13, 14] is invariant under DDT on the uniform (linear) and exponential (q-linear) lattices. For the terminology of lattices in connection to OP, see [17]. This allows for a new characterization of these polynomials as self-similar systems arising from the general q-periodic reductions of the DDT, which is the main result of the present paper.

2. Spectral transformations for orthogonal polynomials

In this section we describe two possible types of DDT for OP having the property that the transformed polynomial has the same degree as the initial one. We also present a set of explicit formulae describing the main features of the transformed OP for both types of transformation.

In what follows we restrict ourselves to the case of monic OP $P_n(x)$ satisfying the recurrence relation

$$P_{n+1}(x) + u_n P_{n-1}(x) + b_n P_n(x) = x P_n(x) \qquad n = 1, 2, \ldots \tag{2.1}$$

and the initial conditions

$$P_0(x) = 1 \qquad P_1(x) = x - b_0. \tag{2.2}$$

By the term 'monic' we mean that the leading term in $P_n(x)$ is x^n. (This clearly follows from (2.1) and (2.2).) If $u_n > 0, n = 1, 2, \ldots,$ and b_n are real for $n = 0, 1, \ldots,$ the polynomials $P_n(x)$ are then orthogonal on the real axis with some positive definite measure $\mathrm{d}\sigma(x)$ (Favard's theorem [3])

$$\int_{a_1}^{a_2} P_n(x) P_m(x)\, \mathrm{d}\sigma(x) = h_n \delta_{nm} \tag{2.3}$$

where $h_0 = 1$, $h_n = u_1 u_2 \ldots u_n > 0$, $n > 0$, are the normalization constants and where the limits of integrations are not *a priori* restricted, $-\infty \leqslant a_1 < a_2 \leqslant \infty$. In principle, the measure may contain a singular continuous part bearing the Cantor set properties which prevents definition of the weight function $w(x)$: $\mathrm{d}\sigma(x) = w(x)\, \mathrm{d}x$. For simplicity of notation, we skip such situations but we allow for the measure to contain discrete jumps, in which case the weight function $w(x)$ comprises a number of Dirac delta-functions.

The functions

$$F_n(x) = \int_{a_1}^{a_2} \frac{P_n(y) w(y)\, \mathrm{d}y}{x - y} \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

define the second linearly independent solution of the recurrence relation (2.1) with the initial condition $F_1(x) = (x - b_0) F_0(x) - 1$. The Stieltjes function

$$F(x) \equiv F_0(x) = \int_{a_1}^{a_2} \frac{w(y)\, \mathrm{d}y}{x - y} \tag{2.5}$$

plays a very important role in the theory of orthogonal polynomials [1, 3]. It represents the generating function of moments c_k:

$$F(x) = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}} \qquad c_k = \int_{a_1}^{a_2} y^k w(y)\, \mathrm{d}y.$$

Quite often in looking for $w(x)$, it proves more convenient or easier to calculate $F(x)$ and to use subsequently the inverse Stieltjes transform to obtain the weight function $w(x)$ [1].

One of the numerous important properties of the Stieltjes function is its representation in the form of a continued fraction

$$F(x) = \cfrac{1}{x - b_0 - \cfrac{u_1}{x - b_1 - \cfrac{u_2}{x - b_2 - \cdots}}}. \tag{2.6}$$

This formula allows one to recover the recurrence coefficients u_n, b_n if $F(x)$ is given. Let us remark that this approach was successfully applied to the characterization of general solutions of discrete integrable systems [4, 18].

The first type of spectral transformations for orthogonal polynomials was proposed by Christoffel in the last century. The corresponding transformed OP are known as kernel polynomials [3]

$$\tilde{P}_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x - \mu} \tag{2.7}$$

where $A_n = P_{n+1}(\mu)/P_n(\mu)$ and μ is some parameter. The recurrence coefficients are transformed into

$$\tilde{u}_n = \frac{u_n A_n}{A_{n-1}} \qquad \tilde{b}_n = b_{n+1} + A_{n+1} - A_n \tag{2.8}$$

and the weight and Stieltjes functions into

$$\tilde{w}(x) = \frac{(x-\mu)w(x)}{b_0 - \mu} \tag{2.9}$$

$$\tilde{F}(x) = \frac{(x-\mu)F(x) - 1}{b_0 - \mu}. \tag{2.10}$$

It is easily seen that the condition $\tilde{w}(x) > 0$ leads to the restriction $\mu \leqslant a_1$ or $\mu \geqslant a_2$, i.e. the auxiliary spectral parameter μ should lie outside the spectral interval. From (2.8) one finds the transformation laws for the normalization constants

$$\tilde{h}_n = h_n A_n (\mu - b_0)^{-1} \qquad n = 1, 2, \ldots, \tilde{h}_0 = h_0 = 1. \tag{2.11}$$

We shall refer to the transformation (2.7) as the Christoffel transform (CT). It is clear from (2.11) that the CT preserves the normalization of the weight function.

Another (more general) type of spectral transformation for OP was considered in detail by Geronimus in 1940 [7, 8] (it was later rediscovered many times in different contexts, see e.g. [9, 14, 26] and references therein). The Geronimus transform (GT) is written in the form

$$\tilde{P}_0 = 1 \qquad \tilde{P}_n(x) = P_n(x) - B_n P_{n-1}(x) \qquad n = 1, 2, \ldots \tag{2.12}$$

where $B_n = \phi_n/\phi_{n-1}$ and the function ϕ_n is a solution of the recurrence relation (2.1) for an auxiliary spectral parameter μ:

$$\phi_n = F_n(\mu) + \beta P_n(\mu) \qquad n = 0, 1, \ldots. \tag{2.13}$$

In general β is arbitrary. However, if one wishes the new polynomials $\tilde{P}_n(x)$ to possess a positive weight function then there are some restrictions on the values of β while the values of μ should be constrained as in the case of the CT.

The recurrence coefficients are transformed according to the formulae

$$\tilde{u}_1 = \frac{\phi_1}{\phi_0^2} \qquad \tilde{u}_n = \frac{u_{n-1}B_n}{B_{n-1}} \qquad n = 2, 3, \ldots$$

$$\tilde{b}_0 = b_0 + \frac{\phi_1}{\phi_0} \qquad \tilde{b}_n = b_n + B_{n+1} - B_n \qquad n = 1, 2, \ldots. \tag{2.14}$$

Note that the apparent mismatch in the expressions (2.14) of $\tilde{u}_n$, $\tilde{b}_n$ for $n = 0, 1$ and the higher n is due to the specific initial conditions (2.2) that determine OP solutions.

The transformation laws for the weight and Stieltjes functions have the form [7]

$$\tilde{w}(x) = \frac{\beta\delta(x-\mu) + w(x)(\mu - x)^{-1}}{\beta + F(\mu)} \tag{2.15}$$

$$\tilde{F}(x) = \frac{F(\mu) + \beta - F(x)}{(x-\mu)(\beta + F(\mu))}. \tag{2.16}$$

It is seen from (2.15) that as a result of the GT the weight function picks up an additional discrete mass at the point $x = \mu$. This is equivalent to the addition of a pole in the Stieltjes function as is seen from (2.16). The normalization constants are transformed as follows

$$\tilde{h}_0 = 1 \qquad \tilde{h}_n = \frac{h_{n-1}\phi_n}{\phi_0\phi_{n-1}} \qquad n = 1, 2, \ldots. \tag{2.17}$$

In contrast to the CT which involves only one parameter μ, the GT depends on two parameters μ and β. Let us denote the Christoffel transformation by $C(\mu)$ and the Geronimus transformation by $G(\mu, \beta)$ (i.e. the action of the operator $C(\mu)$ on orthogonal polynomial $P_n(x)$ gives $\tilde{P}_n(x)$ as in (2.7), etc). It is easily seen that for different values of μ these transformations commute with one another: $C(\mu_1)G(\mu_2, \beta) = G(\mu_2, \beta)C(\mu_1)$. However, for the same parameter μ we have the following relations

$$C(\mu)G(\mu, \beta) = 1 \qquad G(\mu, \beta)C(\mu) = U(\mu, \beta) \tag{2.18}$$

where $U(\mu, \beta)$ is the so-called Uvarov transformation [27] which has the effect of adding a single mass to the weight function at the point $x = \mu$:

$$\tilde{w}(x) = \frac{w(x) + \beta(\mu - b_0)\delta(x-\mu)}{1 + \beta(\mu - b_0)}. \tag{2.19}$$

The transformation law for $F(x)$ is obvious from (2.19). Let us remark that transformation formulae for the Stieltjes function are not sensitive to the presence of the singular continuous part in the orthogonality measure.

Equations (2.18) follow easily from (2.9) and (2.15). The important relations described above were written by Geronimus in a somewhat different (but equivalent) form in [8]. Note that, in fact, the Uvarov transformation is contained in an implicit form in [8]. Combining several transformations $C(\mu)$ and $G(\mu, \beta)$ with different values of the parameters μ and β, we can construct spectral transformations for OP generated by linear difference operators of arbitrary order. These transformations were described explicitly in [8] in a determinant form. This form was also discovered later in many other papers. Below we shall refer sometimes to CT and GT as DDT assuming that they perform a map of OP onto OP.

Consider spectral transformations composed from K different CT with parameters μ_i, $i = 1, 2, \ldots, K$ and J different GT with parameters ν_i, β_i, $i = 1, 2, \ldots, J$. Assume that $\mu_k \neq \nu_i$ for all values of i, k. This condition guarantees commutativity of all transformations. Hence we can first perform the CT and then the GT. Let $F(x)$ be the Stieltjes function for the initial OP. It can be shown by induction that the transformed Stieltjes function is

$$F_{KJ}(x) = \frac{\sigma_{KJ}(x-\mu_1)(x-\mu_2)\ldots(x-\mu_K)F(x) + R_{KJ}(x)}{(x-\nu_1)(x-\nu_2)\ldots(x-\nu_J)} \tag{2.20}$$

where σ_{KJ} is a constant, and $R_{KJ}(x)$ is some polynomial the degree of which is equal to $\max(K-1, J-1)$. The corresponding monic orthogonal polynomials $P_n^{(KJ)}(x)$ can be explicitly presented in the form

$$
(x-\mu_1)(x-\mu_2)\ldots(x-\mu_K)P_n^{(KJ)}(x)
= D_n^{-1}\begin{vmatrix}
P_{n-J}(x) & P_{n-J+1}(x) & \ldots & P_{n+K}(x)\\
P_{n-J}(\mu_1) & P_{n-J+1}(\mu_1) & \ldots & P_{n+K}(\mu_1)\\
\ldots & \ldots & \ldots & \ldots\\
P_{n-J}(\mu_K) & P_{n-J+1}(\mu_K) & \ldots & P_{n+K}(\mu_K)\\
\phi_{n-J}(\nu_1;\beta_1) & \phi_{n-J+1}(\nu_1;\beta_1) & \ldots & \phi_{n+K}(\nu_1;\beta_1)\\
\ldots & \ldots & \ldots & \ldots\\
\phi_{n-J}(\nu_J;\beta_J) & \phi_{n-J+1}(\nu_J;\beta_J) & \ldots & \phi_{n+K}(\nu_J;\beta_J)
\end{vmatrix} \tag{2.21}
$$

where

$$
D_n=\begin{vmatrix}
P_{n-J}(\mu_1) & P_{n-J+1}(\mu_1) & \ldots & P_{n+K-1}(\mu_1)\\
\ldots & \ldots & \ldots & \ldots\\
P_{n-J}(\mu_K) & P_{n-J+1}(\mu_K) & \ldots & P_{n+K-1}(\mu_K)\\
\phi_{n-J}(\nu_1;\beta_1) & \phi_{n-J+1}(\nu_1;\beta_1) & \ldots & \phi_{n+K-1}(\nu_1;\beta_1)\\
\ldots & \ldots & \ldots & \ldots\\
\phi_{n-J}(\nu_J;\beta_J) & \phi_{n-J+1}(\nu_J;\beta_J) & \ldots & \phi_{n+K-1}(\nu_J;\beta_J)
\end{vmatrix}. \tag{2.22}
$$

Recall that $\phi_n(z;\beta)=F_n(z)+\beta P_n(z)$ is an arbitrary solution of the recurrence equation $\phi_{n+1}+b_n\phi_n+u_n\phi_{n-1}=z\phi_n$.

The case of coinciding parameters $\mu_j=\mu_k$ or (and) $\nu_j=\nu_k$, $\beta_j=\beta_k$ for some j,k can be considered analogously and leads to the replacement of entries in the corresponding strings in (2.21) by their derivatives. We do not analyse this (special) situation here. The formula (2.21) was obtained by Uvarov [27] for the special case $\beta_j=0$, $j=1,2,\ldots,J$, when no additional discrete masses appear in the transformed measure.

It is seen from (2.20) that the Stieltjes function $\tilde{F}(x)$ obtained from $F(x)$ through a finite number of arbitrary CT and GT can be presented in the form

$$
\tilde{F}(x)=R_1(x)F(x)+R_2(x) \tag{2.23}
$$

where $R_1(x)$ and $R_2(x)$ are some rational functions in the argument x. One can also prove the converse statement (see, e.g., [29])

Proposition 1. If two Stieltjes functions $\tilde{F}(x)$ and $F(x)$ are related to each other by the linear transformation (2.23) with rational coefficients $R_1(x)$ and $R_2(x)$ then the corresponding OP $\tilde{P}_n(x)$ and $P_n(x)$ are obtained from each other through finite numbers of CT and GT.

This proposition will be used in the next section to characterize semi-classical OP on uniform and exponential lattices. Actually, Maroni proposed in [15] another equivalent form of the OP transformation corresponding to the general linear transformation of the Stieltjes function

$$
B(x)P_n^{(KJ)}(x)=\sum_{k=n-J}^{n+K}\xi_{n,k}P_k(x) \tag{2.24}
$$

where $B(x)$ is a polynomial of the Kth degree and $\xi_{n,k}$ are some numbers (not depending on x). It is obvious that the determinant formula (2.22) can be rewritten in the form (2.24). It was shown in [15] that, conversely, if there is a relation of the form (2.24) between two OP (with some polynomial $B(x)$ and some coefficients $\xi_{n,k}$) then the polynomials $P_n^{(KJ)}$ can be obtained from $P_n(x)$ by applying K Christoffel and J Geronimus transforms. (Strictly speaking, Maroni proved this statement in a slightly different but equivalent formulation, without referring to the CT and GT.)

It is worth stressing that the CT and GT are the only first-order spectral transformations that transform OP into OP. Here 'first order' means that the transformations involve only two polynomials $P_{n+j}(x)$ and $P_{n+j-1}(x)$, where j is some integer. For the CT, $j = 1$ and for the GT $j = 0$. There are no non-trivial maps of OP into OP for other j. A more essential remark is that the second-order DDTs that map OP into OP with positive measures are richer than the first-order ones. Indeed, the choice $a_1 < \mu < a_2$ spoils the positivity of the weight functions for both CT and GT, but the second-order DDT may cure this problem in such a way that the measure will be positive while a mass point is inserted into the continuous spectrum [26]. We shall not consider these more complicated situations here.

3. Quasi-periodic reductions of spectral transformations and semi-classical orthogonal polynomials

Obviously the iterations of CT and GT can be put into the chain form (1.6). However, this would create a mismatch in notation in the context of OP and remove distinguishing properties of the taken spectral transformations. Therefore in the discussion of self-similar solutions of (1.6) given below, we use notation different from [23].

In this section we consider (quasi-)periodic similarity reductions (or closures) of the chains of spectral transformations for OP. These closures lead either to OP which are orthogonal on finite sets of points (in the case of purely periodic closure), or to the so-called semi-classical OP on a uniform lattice which are non-trivial generalizations of the corresponding classical OP, i.e. of the Charlier, Krawtchouk, Meixner and Hahn polynomials.

In analogy with the analysis of the previous section, consider the action of N DDTs on a set of OP. We assume that $N = K + J$ where K is the number of CTs and J is the number of GTs. Denote by $u_n^{(N)}$ and $b_n^{(N)}$ the recurrence coefficients obtained as a result of this procedure (see the transformation formulae (2.8) and (2.14)). The quasi-periodic closure condition means that [23]

$$u_n^{(N)} = u_n \qquad b_n^{(N)} = b_n + \alpha \tag{3.1}$$

where α is some constant. The condition (3.1) can be considered as a discrete version of the quasi-periodic closure of the chain of standard Darboux transformations for the ordinary Schrödinger equation which was analysed by Veselov and Shabat in [28]. Note that the meaning of the term 'quasi-periodic' in our context does not coincide with that for almost periodic functions. In the notation of equations (1.6) it is clear that we deal with functions of two discrete variables j and n such that the shift of j by N does not affect u_n^j but forces b_n^j to acquire a homogeneous shift by α.

Consider first the purely periodic closure, i.e. the case $\alpha = 0$. Then it is clear that after N DDTs the orthogonal polynomials return to their initial form $P_n^{(N)}(x) = P_n(x)$, hence $F^{(N)}(x) = F(x)$. Using (2.20) we get

$$F(x) = \frac{R_{KJ}(x)}{Q_{KJ}(x)} \tag{3.2}$$

where $R_{KJ}(x)$ is a polynomial appearing in (2.20) and $Q_{KJ}(x)$ is

$$Q_{KJ}(x) = (x - \nu_1)\ldots(x - \nu_J) - \sigma_{KJ}(x - \mu_1)\ldots(x - \mu_K). \tag{3.3}$$

It is clear from (3.3) that the degree of the polynomial $Q_{KJ}(x)$ exceeds the degree of $R_{KJ}(x)$ by one. Hence the Stieltjes function admits an expansion in terms of simple fractions

$$F(x) = \sum_{k=1}^{L} \frac{M_k}{x - x_k} \tag{3.4}$$

where x_k are the roots of the polynomial $Q_{KJ}(x)$ and $L = \max(K, J)$. The parameters M_k are called Christoffel numbers [3]. They play the role of discrete masses located at the points x_k since the corresponding weight function has the form

$$w(x) = \sum_{k=1}^{L} M_k \delta(x - x_k). \tag{3.5}$$

Thus in the case of the purely periodic closure, we obtain OP which are orthogonal on the finite set of points x_k with the masses M_k. It is clear that all finite-dimensional orthogonal polynomials can be obtained by such a procedure.

Consider now the less trivial quasi-periodic closure, i.e. the case $\alpha \neq 0$. Without loss of generality we can set $\alpha = 1$ by appropriate rescaling of the argument x. We then have for the transformed OP $P_n^{(N)}(x) = P_n(x-1)$ and hence $F^{(N)}(x) = F(x-1)$. Using (2.20) we get the following finite-difference equation for the Stieltjes function:

$$F(x-1) = \frac{\sigma_{KJ}(x-\mu_1)\dots(x-\mu_K)F(x) + R_{KJ}(x)}{(x-\nu_1)\dots(x-\nu_J)}. \tag{3.6}$$

We will seek solutions of this equation in the form

$$F(x) = \sum_{k=0}^{\infty} \frac{M_k}{x - x_k} \tag{3.7}$$

with some unknown x_k and M_k. Again, the Christoffel numbers M_k play the role of the discrete masses located at the points x_k. However, there are now in general infinitely many of them. Let us demand that the spectral points x_k are bounded from below. Since ν_i cannot coincide with any x_k this condition requires $\beta_i = 0$ (for generic values of μ_i, ν_i). Substituting (3.7) into (3.6), we see that under the condition $\mu_i - \mu_j \neq m$, $i \neq j$, m-integer, the x_k form up to K arithmetic progressions

$$x_k^{(m)} = a_m + k \qquad m = 1, 2, \dots, L \leqslant K \qquad k = 0, 1, \dots, \infty \tag{3.8}$$

where a_m are some constants. Appearance of the discrete spectrum as a superposition of up to N arithmetic progressions is easily seen from the symmetry algebra of the taken models [24], but one has to know a split of N DDTs into K CTs and J GTs in order to determine the possible number of equidistant series in the spectrum more precisely.

Using (3.8), we can rewrite expression (3.7) for the Stieltjes function in the form

$$F(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{L} \frac{M_k^{(m)}}{x - a_m - k}. \tag{3.9}$$

From the consistency condition of this ansatz with (3.6) we find that a_m should coincide with some of μ_m, say

$$a_m = \mu_m \qquad m = 1, 2, \dots, L. \tag{3.10}$$

For the masses we obtain the equation

$$\frac{M_{k+1}^{(m)}}{M_k^{(m)}} = \sigma_{KJ}^{-1} \frac{\prod_{s=1}^{J}(1 + k + \mu_m - \nu_s)}{\prod_{s=1}^{K}(1 + k + \mu_m - \mu_s)} \tag{3.11}$$

which allows us to calculate them explicitly

$$M_k^{(m)} = M_0^{(m)} \sigma_{KJ}^{-k} \frac{\prod_{s=1}^{J} (\xi_{ms})_k}{\prod_{s=1}^{K} (\eta_{ms})_k} \qquad k = 1, 2, \ldots, \infty \tag{3.12}$$

where $\xi_{ms} = 1 + \mu_m - \nu_s$, $\eta_{ms} = 1 + \mu_m - \mu_s$ and $(a)_k = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol. The values of the 'initial' masses $M_0^{(m)}$ are arbitrary with the normalization constraint $\sum_{m=1}^{L} \sum_{k=0}^{\infty} M_k^{(m)} = 1$ as the only restriction.

For each mth branch of the spectrum the sequence $M_k^{(m)}$ is nothing else than the generalized hypergeometric distribution. Hence, we obtain that the weight functions of the OP obtained through quasi-periodic closure of the DDT form a set of L generalized hypergeometric distributions. The cases $\mu_i - \mu_j = m$, m-integer, for some i, j, are degenerate, for example one may get a truncation of arithmetic progressions, there appear ambiguities in the counting of spectral branches, form of the symmetry algebra, etc (see [22] for the differential Schrödinger equation examples). If $K = 0$ then there are no solutions of the form (3.7).

In a similar way, one can assume that the spectral points x_k are bounded from above. For the existence of solutions of the form (3.7) one has to take $\beta_i \neq 0$. Then, under the conditions $\nu_i - \nu_j \neq m$, one finds x_k as a superposition of up to J arithmetic progressions

$$x_k^{(m)} = \nu_m - 1 - k \qquad m = 1, 2, \ldots, L \leqslant J \qquad k = 0, 1, \ldots. \tag{3.13}$$

The systems of OP with spectral points bounded from above can be obtained from those with spectral points bounded from below by the reflection transformation: $P_n(x) \to (-1)^n P_n(-x)$. If one abandons semi-boundedness of the spectrum, then one obtains formally that $F(x)$ is composed from up to $J + K$ arithmetic progressions of poles. Convergence issues are important for all these functions. The problem of classification of admissible solutions of (3.6), in particular of those related to positive measures, lies beyond the scope of the present work.

Consider the simplest systems arising in this approach. When $K = 1$, $J = 0$ (i.e. when we have only one Christoffel transformation) formula (3.12) yields the masses

$$M_k = \mathrm{e}^{-\gamma} \frac{\gamma^k}{k!} \tag{3.14}$$

located at the points of the single arithmetic progression

$$x_k = \mu + k \qquad k = 0, 1, \ldots, \infty \tag{3.15}$$

with an arbitrary parameter μ. The Poisson distribution (3.14) defines the Charlier polynomials [3]. For $K = 0$, $J = 1$ the same Charlier polynomials arise.

For the case $K = J = 1$, i.e. CT and GT with spectral parameters $\mu \neq \nu$ and $\beta = 0$, we find the masses

$$M_k = M_0 \sigma^{-k} \frac{(1 + \mu - \nu)_k}{k!} \tag{3.16}$$

located at the same points (3.15). For $\sigma > 0$, $\mu > \nu$ we get the Meixner polynomials, for $\sigma < 0$, $\nu - \mu = 2, 3, \ldots$ the Krawtchouk polynomials. Already for the case $K = 2$, $J = 0$ (or $J = 2$, $K = 0$), the corresponding polynomials are not classical since their recurrence coefficients cannot be expressed in terms of elementary functions. There remains only one case $J = K = 2$ for which the recurrence coefficients are elementary functions [24] and it corresponds to the classical Hahn polynomials. Following the lines of [28] it is natural to expect that the equations for the recurrence coefficients u_n and b_n associated with higher values of J and K can be expressed in terms of the higher-order discrete Painlevé transcendents [23]. Unfortunately the full analytical theory of these functions has not yet been developed; for some steps in this direction see, e.g., [5, 12].

What class of OP is defined by the quasi-periodic closure (3.1) of the chain of CT and GT? In [11, 13] Magnus has considered a very general class of orthogonal polynomials, called Laguerre–Hahn polynomials, for which the Stieltjes function satisfies by definition the following nonlinear difference equation (generalized Riccati equation):

$$a(x)\frac{F(y_2)-F(y_1)}{y_2-y_1} = b(x)F(y_1)F(y_2)+c(x)(F(y_2)+F(y_1))+d(x) \tag{3.17}$$

where $a(x)$, $b(x)$, $c(x)$, $d(x)$ are arbitrary polynomials in x preserving the correct asymptotic behaviour $F(x)\to 1/x+\mathrm{O}(1/x^2)$ of the Stieltjes function at $x\to\infty$. In general, the corresponding OP live on non-uniform lattice defined by the functions $y_1(x)$ and $y_2(x)$ such that [11]: $y_1(x)+y_2(x) =$ polynomial of degree 1, $y_1(x)y_2(x) =$ polynomial of the degree $\leqslant 2$. On the left-hand side of (3.17) one finds a definition of the divided difference operator (without the factor $a(x)$) that possesses an important property—it maps any polynomial of degree n onto a polynomial of degree $n-1$.

In the case $b(x)=0$ the equation (3.17) becomes a linear equation which by definition corresponds to the so-called semi-classical orthogonal polynomials (SCOP) [11, 14]. The fact that discrete masses of general SCOP appear in correspondence with the lattices parametrizing the polynomials' argument was noticed in [11, 13]. The 'classical' OP form a subclass of SCOP characterized by the condition that the degrees of the polynomials $a(x)$, $c(x)$ and $d(x)$ are equal to 2, 1 and 0, respectively. Equation (3.17) has a continuous limit leading to the differential Riccati equation; for some examples of the corresponding systems see, for example, [2].

Let us focus on the SCOP defined on the uniform lattice with unit step, i.e. with $y_1(x)=x$, $y_2(x)=x-1$. The Stieltjes function for such polynomials obeys the equation

$$F(x-1) = D(x)F(x)+E(x) \tag{3.18}$$

where $D(x)$ and $E(x)$ are some rational functions in the argument x. Now briefly consider the case of 'classical' polynomials on a uniform lattice. From the above-mentioned definition it follows that

$$D(x) = \frac{(x-\alpha_1)(x-\alpha_2)}{(x-\beta_1)(x-\beta_2)} \qquad E(x) = \frac{\delta}{(x-\beta_1)(x-\beta_2)}$$

where $\alpha_1,\ldots,\delta$ are some parameters. From (3.6) it is easy to see that this case corresponds to a special subcase of the $N=4$ quasi-periodic closure of the DDT consisting of two CTs and two GTs. Such a closure condition characterizes the ordinary Hahn polynomials [24].

It was shown [13] that the defining property (3.17) for the Stieltjes function $F(x)$ is equivalent to the existence of some difference–difference relations for the corresponding OP. For the uniform lattice the property (3.18) is equivalent to the following.

Proposition 2. The orthogonal polynomials $P_n(x)$ are semi-classical (on the uniform lattice) if and only if they obey the following first-order difference–difference relations

$$\begin{aligned} P_n(x-1) &= V_n(x)P_n(x)+W_n(x)P_{n-1}(x) \\ P_{n-1}(x-1) &= Y_n(x)P_n(x)+Z_n(x)P_{n-1}(x) \end{aligned} \tag{3.19}$$

where $V_n(x)$, $W_n(x)$, $Y_n(x)$, $Z_n(x)$ are some rational functions in x such that their degrees do not depend on the number n (i.e. all dependence on n is contained in the coefficients of the corresponding polynomials that enter these rational functions).

It is convenient to rewrite (3.19) in the matrix form:

$$\begin{pmatrix} P_n(x-1) \\ P_{n-1}(x-1) \end{pmatrix} = U_n(x) \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} \tag{3.20}$$

where the matrix $U_n(x)$ is

$$U_n(x) = \begin{pmatrix} V_n(x) & W_n(x) \\ Y_n(x) & Z_n(x) \end{pmatrix}. \tag{3.21}$$

As a simple consequence of (3.19) we see that the semi-classical polynomials obey the second-order difference equation

$$\begin{aligned} &W_n(x+1)P_n(x-1) + W_n(x)\Delta(x+1)P_n(x+1) \\ &\qquad = (V_n(x)W_n(x+1) + W_n(x)Z_n(x+1))P_n(x) \end{aligned} \tag{3.22}$$

where $\Delta(x) = V_n(x)Z_n(x) - Y_n(x)W_n(x)$ is the determinant of the matrix $U_n(x)$. It should be noted that for the classical orthogonal polynomials on the uniform lattice (such as Charlier, Meixner, Krawtchouk and Hahn polynomials) the coefficients in the difference equation (3.22) do not depend on n except for the diagonal term which contains the spectral parameter.

From (3.18) and (3.6) we arrive at the following.

Proposition 3. Polynomials obtained by the quasi-periodic reduction of the chain of Christoffel and Geronimus transformations belong to the class of SCOP on the uniform lattice.

Using the explicit formulae (2.10) and (2.16) it is fairly easy to see that the class of SCOP is invariant under the CT and GT, i.e. these transformations preserve the semi-classical nature of polynomials.

The main statement of this section is the converse of proposition 3.

Proposition 4. All semi-classical orthogonal polynomials on the uniform lattice can be characterized as resulting from the quasi-periodic reduction of the chain of Christoffel and Geronimus spectral transformations.

Indeed, after a number of CTs and GTs the Stieltjes function of the corresponding OP will be

$$\tilde{F}(x) = R_1(x)F(x) + R_2(x) \tag{3.23}$$

where $R_1(x)$ and $R_2(x)$ are some rational functions. From proposition 1 we know on the one hand that the relation (3.23) implies that the corresponding OP are related to each other through a finite number of CTs and GTs. On the other hand, since the reduction is quasi-periodic, $\tilde{F}(x) = F(x-1)$. Hence we see that the equation

$$F(x-1) = R_1(x)F(x) + R_2(x) \tag{3.24}$$

simultaneously leads to SCOP on the uniform lattice (see (3.18)) and OP resulting from the quasi-periodic reduction (3.1). Thus these two sets of OP coincide.

From the Maroni equivalent form of presentation of the finite chain of spectral transformations (2.24) one can deduce another characterization of SCOP on the uniform lattice. Namely, the orthogonal polynomials $P_n(x)$ are SCOP on the uniform lattice if and only if they obey the equation

$$B(x)P_n(x-1) = \sum_{k=n-J}^{n+K} \xi_{n,k} P_k(x) \tag{3.25}$$

where J and K are some positive integers, $B(x)$ is some polynomial of the degree K and $\xi_{n,k}$ are some constants. This statement is obtained straightforwardly by combining proposition 4 and the property (2.24).

Let us recall that quasi-periodic reductions of the standard Darboux transformation chain to a finite set of nonlinear differential equations has been considered in [28]. This reduction defined Schrödinger operators whose formal spectra consist of N arithmetic progressions and whose symmetry operators are given by a finite-order differential operator. Proposition 4 can be considered as a solution of the discrete analogue of this problem when the Schrödinger operator is replaced by a Jacobi matrix describing a three-term recurrence relation for OP. This means that recurrence coefficients of SCOP on the uniform lattice are exact orthogonal polynomial analogues of the harmonic oscillator potential and of the more complicated potentials related to some Painlevé functions and their higher-order analogues. Note that there are more general Schrödinger operators with equidistant spectra [21, 22], whose symmetries are generated by a differential–difference operator. Their discrete analogues are discussed in [23] and section 5.

Returning to the OP we remark that starting from the ordinary classical polynomials on the uniform lattice one can construct many explicit examples of semi-classical polynomials simply by applying an arbitrary number of CTs or GTs. One of the interesting consequences of this observation is the following. As was noted, the Uvarov transformation (i.e. the addition of a single discrete mass to the weight function) is a combination of CTs and GTs. Hence all polynomials whose weight function differs from the classical one by masses added at arbitrary locations are semi-classical and hence obey a second-order difference equation.

Let us derive the second-order difference equation for the transformed polynomials from a given initial equation. Indeed, let the initial SCOP $P_n(x)$ have recurrence functions $V_n(x), \ldots, Z_n(x)$ in the system (3.20) and let $\tilde{P}_n(x)$ be the OP obtained from $P_n(x)$ by a finite number of DDTs. As was shown, the $\tilde{P}_n(x)$ are also SCOP and hence by the theorem of Magnus they should also obey the system (3.20)

$$\begin{pmatrix} \tilde{P}_n(x-1) \\ \tilde{P}_{n-1}(x-1) \end{pmatrix} = \tilde{U}_n(x) \begin{pmatrix} \tilde{P}_n(x) \\ \tilde{P}_{n-1}(x) \end{pmatrix} \tag{3.26}$$

where the matrix $\tilde{U}_n(x)$ contains some new rational functions $\tilde{V}_n(x), \ldots, \tilde{Z}_n(x)$. Now, from (2.21) one can always write

$$\begin{pmatrix} \tilde{P}_n(x) \\ \tilde{P}_{n-1}(x) \end{pmatrix} = M_n(x) \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} \tag{3.27}$$

where the matrix

$$M_n(x) = \begin{pmatrix} A_n(x) & B_n(x) \\ C_n(x) & D_n(x) \end{pmatrix}$$

and $A_n(x)$, $B_n(x)$, $C_n(x)$, $D_n(x)$ are some rational functions. Comparing (3.20), (3.26) and (3.27) we easily find that the transformed OP satisfy the difference–difference relations (3.26), where the matrix $\tilde{U}_n(x)$ is

$$\tilde{U}_n(x) = M_n(x-1)U_n(x)M_n^{-1}(x). \tag{3.28}$$

From (3.26) we then easily reconstruct the second-order difference equation (3.22) for the polynomials $\tilde{P}_n(x)$ with $V_n(x), \ldots, Z_n(x)$ replaced by $\tilde{V}_n(x), \ldots, \tilde{Z}_n(x)$.

Thus, in order to construct the difference equations for the transformed OP $\tilde{P}_n(x)$, it is sufficient to know explicitly two matrices: $M_n(x)$ which results from a number of CTs and GTs and $U_n(x)$ which determines the difference–difference relation for the polynomials $P_n(x)$. The evaluation of both these matrices can easily be made algorithmic.

4. Self-similar reductions and semi-classical polynomials on the q-linear lattice

In this section we briefly analyse another possible reduction of the chain of spectral transformations—a simple version of the q-periodic closure. Such self-similar reductions were first considered for the chain of Darboux transformations associated with the standard Schrödinger equation (see [22] and references therein). For the discrete Schrödinger equation, they were considered first in [23].

The q-periodic closure condition of the chain of DDTs is defined (instead of (3.1)) by the conditions

$$u_n^{(N)} = q^2 u_n \qquad b_n^{(N)} = q b_n \tag{4.1}$$

where q is some real parameter. As in the previous section we assume that $N = K + J$.

It is easily seen from (4.1) that after N DDTs the polynomials $P^{(N)}(x)$ are expressed in terms of the initial polynomials by the simple formula

$$P_n^{(N)}(x) = q^n P_n(q^{-1}x). \tag{4.2}$$

Hence the transformed Stieltjes function has the form

$$F^{(N)}(z) = q^{-1} F(q^{-1}z). \tag{4.3}$$

In place of (3.6) we thus get the following equation for the Stieltjes function:

$$q^{-1} F(q^{-1}x) = \frac{\sigma_{KJ}(x-\mu_1)\dots(x-\mu_K)F(x) + R_{KJ}(x)}{(x-\nu_1)\dots(x-\nu_J)}. \tag{4.4}$$

Repeating the reasoning of the previous section, we see that in the ansatz (3.7) the spectral points x_k form now, under the conditions $\beta_i = 0$ and $\mu_i/\mu_j \neq q^m$, $i \neq j$, for some integer m, a set of up to K geometric progressions:

$$x_k^{(m)} = \mu_m q^k \qquad k = 0, 1, \dots, m = 1, 2, \dots, L \leqslant K. \tag{4.5}$$

For the corresponding masses $M_k^{(m)}$, we have the equations

$$\frac{M_k^{(m)}}{M_{k-1}^{(m)}} = \sigma_{KJ}^{-1} \frac{\prod_{s=1}^{J}(\mu_m q^k - \nu_s)}{\prod_{s=1}^{K}(\mu_m q^k - \mu_s)} \qquad k = 1, 2, \dots \tag{4.6}$$

i.e. they form generalized q-hypergeometric distributions.

We will not consider particular examples here and discuss only the general case. Recall that the semi-classical orthogonal polynomials on the exponential or q-linear lattice are defined through the following q-difference equation for their Stieltjes function [11, 13]

$$F(x/q) = R_1(x)F(x) + R_2(x) \tag{4.7}$$

where $R_{1,2}(x)$ are some rational functions that guarantee that $F(x)$ asymptotically behaves as $1/x + \mathrm{O}(1/x^2)$ for $x \to \infty$. Arguing as in the proof of proposition 4 we arrive at the following statement.

Proposition 5. General semi-classical orthogonal polynomials on the q-linear lattice are obtained under the q-periodic closure condition (4.1).

Note that a slightly more general closure condition

$$u_n^{(N)} = q^2 u_n \qquad b_n^{(N)} = q b_n + \alpha \tag{4.8}$$

where α is some constant, can be reduced (for $q \neq 1$) to the simple q-periodic one (4.1) by a shift of the recurrence coefficients $b_n \to b_n + \alpha/(1-q)$.

5. General self-similar reductions and Laguerre–Hahn polynomials

In this section we consider more general self-similarity closures when the indices of recurrence coefficients are shifted after a chain of DDTs. Such closures were considered first for the ordinary Schrödinger equation [21] and in the discrete case they appeared in [23]. We show that these generalized reductions of spectral transformations lead to a much wider class of orthogonal polynomials known as the Laguerre–Hahn polynomials [11, 13] on the linear and q-linear lattices.

Let us start with the uniform or linear lattice. The following closure conditions take place:

$$u_{n+k}^{(N)} = u_{n+m} \qquad n = 1, 2, 3, \ldots$$
$$b_{n+k}^{(N)} = b_{n+m} + \alpha \qquad n = 0, 1, 2, \ldots \tag{5.1}$$

where the integer parameters k and m are both positive (or zero). For $N = \alpha = 0$ and $k \neq m$ we get a periodicity condition for u_n, b_n with $n > \min\{k, m\}$. These constraints include as special cases the forward associative transformations ($k = 0$), backward associative transformations ($m = 0$) and transformations to the co-recursive polynomials ($k = m$) [3]. For $k \neq 0$ there is some freedom in the choice of the first coefficients $u_1^{(N)}, u_2^{(N)}, \ldots, u_k^{(N)}$ and $b_0^{(N)}, b_1^{(N)}, \ldots, b_{k-1}^{(N)}$, which are thus arbitrary parameters. Note that for the discrete Schrödinger equations on the full lattice $n \in \mathbb{Z}$, all these cases are equivalent.

Consider first the case when $\alpha = 0$. After N DDTs we get

$$P_n^{(N)}(x; k) = P_n(x; m) \tag{5.2}$$

where $P_n(x; m)$ denotes the so-called m-associated polynomials satisfying the recurrence relation

$$P_{n+1}(x; m) + u_{n+m} P_{n-1}(x; m) + b_{n+m}(x) P_n(x; m) = x P_n(x; m) \tag{5.3}$$

with the initial conditions

$$P_0(x; m) = 1 \qquad P_1(x; m) = x - b_m. \tag{5.4}$$

The Stieltjes function for the m-associated polynomials is found to be [14, 20] (the simplest case was discussed already in [7])

$$F(x; m) = \frac{A'(x)F(x) + B'(x)}{C'(x)F(x) + D'(x)} \tag{5.5}$$

where $F(x) \equiv F(x; 0)$ and $A'(x)$, $B'(x)$, $C'(x)$, $D'(x)$ are polynomials satisfying the condition that $F(x; m) \propto 1/x$ for $x \to \infty$ and

$$A'(x)D'(x) - B'(x)C'(x) = \text{constant}. \tag{5.6}$$

Moreover, it can be shown [20, 29] that any transformation of the Stieltjes function of the form (5.5) with the condition (5.6) is equivalent to passing to some associated polynomials.

Since the whole spectral information is contained in $F(x)$, we see that there is a well defined correspondence between the spectral properties of the initial and m-associated systems corresponding to (5.5), i.e. we have a spectral transformation in the sense given in the introduction. The intertwining relations between the corresponding Jacobi matrices L and $\tilde{L}$ are obvious $DL = \tilde{L}D$, $D\psi_n = \psi_{n+m}$, but these do not map OP onto OP, i.e. a subtler analysis is thus called for. Note that currently the cases of non-integer shifts in (5.1) are not tractable in the described way.

From (5.1) ($\alpha = 0$), the closure condition for the Stieltjes function can be written in the form

$$\frac{\sigma_{KJ}(x-\mu_1)\dots(x-\mu_K)F(x;k)+R_{KJ}(x)}{(x-\nu_1)\dots(x-\nu_J)} = F(x;m). \tag{5.7}$$

Using (5.5) one can rewrite (5.7) in the form

$$S(x)F^2(x)+T(x)F(x)+U(x)=0 \tag{5.8}$$

where $S(x)$, $T(x)$ and $U(x)$ are some polynomials. This equation is known to define the Stieltjes function for orthogonal polynomials of the 'second-degree form' [15]. Such polynomials generalize Akhiezer polynomials orthogonal on several intervals; a large class of them was characterized by Peherstorfer in [19].

Consider now the more complicated case $\alpha \neq 0$. It is convenient to normalize $\alpha = 1$, i.e. to put

$$P_n^{(N)}(x;k) = P_n(x-1;m).$$

It is then easy to see that we have the following condition

$$\frac{\sigma_{KJ}(x-\mu_1)\dots(x-\mu_K)F(x;k)+R_{KJ}(x)}{(x-\nu_1)\dots(x-\nu_J)} = F(x-1;m). \tag{5.9}$$

By the application of relations (5.5), equation (5.9) can be reduced to the difference Riccati equation for the Stieltjes function

$$F(x-1) = \frac{A(x)F(x)+B(x)}{C(x)F(x)+D(x)} \tag{5.10}$$

with some polynomial coefficients $A(x)$, $B(x)$, $C(x)$ and $D(x)$. It turns out that equation (5.10) coincides with the one that defines the Laguerre–Hahn polynomials on the uniform lattice (3.17). Some properties of these polynomials were discussed in [11, 13]. Unfortunately, the analytical theory of these polynomials is in its infancy and many of their characteristic features are not yet recognized.

One can completely analogously consider the generalized q-periodic closure condition

$$u_{n+k}^{(N)} = q^2 u_{n+m} \qquad b_{n+k}^{(N)} = q b_{n+m} \tag{5.11}$$

yielding $P_n^{(N)}(x;k) = q^n P_n(q^{-1}x;m)$, or

$$\frac{\sigma_{KJ}(x-\mu_1)\dots(x-\mu_K)F(x;k)+R_{KJ}(x)}{(x-\nu_1)\dots(x-\nu_J)} = q^{-1}F(q^{-1}x;m). \tag{5.12}$$

This condition leads to the Laguerre–Hahn polynomials on the q-linear lattice with the Stieltjes function determined from the equation

$$F(q^{-1}x) = \frac{A(x)F(x)+B(x)}{C(x)F(x)+D(x)} \tag{5.13}$$

for some polynomial $A(x)$, $B(x)$, $C(x)$ and $D(x)$.

The following statement can be considered as a fundamental contribution to the understanding of the origin of the Laguerre–Hahn polynomials.

Proposition 6. All Laguerre–Hahn polynomials on the linear and q-linear lattices are obtained under the generalized closure conditions (5.1) and (5.11), respectively.

Indeed, using the results of [20, 29] one can see that for general polynomials $A(x)$, $B(x)$, $C(x)$ and $D(x)$ preserving leading asymptotics of $F(x)$ the right-hand side of (5.10) can be represented as an iteration of a finite number of Christoffel, Geronimus and associated polynomial transformations for the function $F(x)$. On the other hand, the left-hand side of (5.10) (i.e. $F(x-1)$) corresponds to the polynomials $P_n(x-1)$ with a shifted argument. Hence these shifted polynomials can be obtained after a finite number of Christoffel and Geronimus transformations followed by a finite number of associated polynomial transformations. However, this is precisely the condition (5.1). It thus characterizes all Laguerre–Hahn polynomials on the uniform lattice. The same arguments are valid for the relation between the q-periodic closure (5.11) and the Laguerre–Hahn polynomials on the exponential lattice. From (2.10) and (2.16) it follows, similar to the situation with SCOP, that both the CT and GT preserve the nature of Laguerre–Hahn polynomials, i.e. this class of OP is invariant under such spectral transformations.

Recurrence coefficients for the Laguerre–Hahn polynomials are determined by some complicated nonlinear mappings representing q-analogues of the discrete Painlevé transcendents and their higher-order generalizations. The main open problem is the description of their analytical and asymptotic properties needed in various physical applications. We would like to finish by remarking that the symmetry algebras formed by the raising and lowering operators of these systems are quite simple [22, 23]. They are determined by three generators satisfying polynomial identities, which in the particular case $N = 2$ (N being the q-period of the chain) define the quantum algebra $U_q(sl_2)$. It should also be pointed out that the applications of discrete exactly solvable potentials arising from (5.11) are not limited to the Laguerre–Hahn polynomials. The systems that go beyond them appear as the full functional solutions of the corresponding discrete Schrödinger equation defined for continuous values of the variable n.

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References

[1] Akhieser N I 1965 *The Classical Moment Problem* (Edinburgh: Oliver and Boyd)

[2] Alaya J and Maroni P 1996 Symmetric Laguerre–Hahn forms of class $s = 1$ *Int. Transf. Spec. Funct.* **4** 301–20

[3] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

[4] Common A K 1992 A solution of the initial problem for half-infinite integrable lattice systems *Inverse Problems* **8** 393–408

[5] Fokas A S, Its A R and Kitaev A V 1991 Discrete Painlevé equations and their appearance in quantum gravity, *Comm. Math. Phys.* **142** 313–44

[6] Gautschi W 1990 Computational aspects of orthogonal polynomials *Orthogonal Polynomials, Theory and Practice* ed P Nevai (Dordrecht: Kluwer)

[7] Geronimus Ya L 1940 On the polynomials orthogonal with respect to a given number sequence *Zap. Mat. Otdel. Khar'kov. Univers. i NII Mat. i Mehan.* **17** 3–18

[8] Geronimus Ya L 1940 On the polynomials orthogonal with respect to a given number sequence and a theorem ed W Hahn *Izv. Akad. Nauk SSSR* **4** 215–28

[9] Iserles A, Koch P E, Nørsett S P and Sanz-Serna J M 1991 On polynomials orthogonal with respect to certain Sobolev inner products *J. Approx. Theory* **65** 151–75

[10] Levi D 1981 Nonlinear differential–difference equations as Bäcklund trasformations *J. Phys. A: Math. Gen.* **14** 1083–98

[11] Magnus A P 1988 Associated Askey–Wilson polynomials as Laguerre–Hahn orthogonal polynomials *Orthogonal Polynomials and their Applications (Lecture Notes in Math. 1329)* ed M Alfaro *et al* (Berlin: Springer) pp 261–78

[12] Magnus A P 1995 Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials *J. Comput. Appl. Math.* **57** 215–37

[13] Magnus A P 1995 Special non uniform lattice (snul) orthogonal polynomials on discrete dense sets of points *J. Comput. Appl. Math.* **65** 253–65

[14] Maroni P 1991 Une théorie algébrique des polynomes orthogonaux. Application aux polynomes orthogonaux semi-classiques. *Orthogonal Polynomials and their Applications (Ann. Comp. Appl. Math. 9)* ed C Brezinski *et al* (Basel: Baltzer) pp 95–130

[15] Maroni P 1995 Tchebychev forms and their perturbed forms as second degree forms *Ann. Numer. Math.* **2** 123–43

[16] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)

[17] Nikiforov A F, Suslov S K and Uvarov V B 1991 *Classical Orthogonal Polynomials of a Discrete Variable* (Berlin: Springer)

[18] Nikishin E M and Sorokin V N 1988 *Rational Approximations and Orthogonality* (Moscow: Nauka)

[19] Peherstorfer F 1990 On Bernstein-Szegö orthogonal polynomials on several intervals *SIAM J. Math. Anal.* **21** 461–82

[20] Peherstorfer F 1992 Finite perturbations of orthogonal polynomials *J. Comput. Appl. Math.* **44** 275–302

[21] Spiridonov V 1992 Deformation of supersymmetric and conformal quantum mechanics through affine transformations *Proc. Int. Workshop on Harmonic Oscillators (College Park, USA, March 1992) (NASA Conf. Publ. 3197)* ed D Han *et al*, pp 93–108

[22] Spiridonov V 1995 Universal superpositions of coherent states and self-similar potentials *Phys. Rev.* A **52** 1909–35

[23] Spiridonov V, Vinet L and Zhedanov A 1993 Difference Schrödinger operators with linear and exponential discrete spectra *Lett. Math. Phys.* **29** 63–73

[24] Spiridonov V, Vinet L and Zhedanov A 1994 Periodic reductions of the factorization chain and the Hahn polynomials *J. Phys. A: Math. Gen.* **27** L669–75

[25] Spiridonov V and Zhedanov A 1995 Discrete reflectionless potentials, quantum algebras and q-orthogonal polynomials *Ann. Phys., NY* **237** 126–46

[26] Spiridonov V and Zhedanov A 1995 Discrete Darboux transformations, the discrete-time Toda lattice and the Askey–Wilson polynomials *Methods Appl. Anal.* **2** 369–98

[27] Uvarov V B 1969 The connection between systems of polynomials that are orthogonal with respect to different distribution functions *Zh. Vychisl. Matem. i Mat. Fiz.* **9** 1253–62

[28] Veselov A and Shabat A 1993 Dressing chain and spectral theory of Schrödinger operator *Funkt. Anal. i Pril.* **27** (2) 1–21

[29] Zhedanov A 1997 Rational spectral transformations and orthogonal polynomials *J. Comput. Appl. Math.* to appear